Problem 4: Multiple Mazes - Solution

We can make a tree out of this figure, by drawing a vertex at every intersection and every exist and connecting when there is a path between two intersections. We see that in this tree, every node either has two children or none. This kind of tree is called a full tree. The amount of leaves of the tree equals the number of exits of the maze.

Now note that for every path, we have either no intersection at a piece of the path, or two ways to go from an intersection. Note that if we are at the first intersection, to our left is again a maze of this form and to the right is a maze of this form, or there is not intersection at the start at all. In the tree structure, we see this as well: from the root, either two full subtrees emerge, or the tree only consists of the root. In total we want to have n exits (or n leaves if you look at it from the tree perspective).

We see that we can calculate the amount of total mazes possible, by looking at the two mazes that appear left and right of the first intersection. If on the left side there are $1 \leq k \leq n-1$ exits, then on the right side of the first intersection there must be $n-k$ exits. Let M_n be the total amount of mazes with n exits. Then we see that:

$$M_n = \sum_{k=1}^{n-1} M_k M_{n-k},$$

as the number of mazes with n exits is determined by the sum over k of the number of options for a maze with k exits on the left multiplied by the number of options for a maze with $n-k$ exits on the right. Also, note that we have that $M_1 = 1$.

Now, using this recurrence relation to calculate the answer is alright, it would pass the first few testcases, but becomes very slow for $n > 15$. It would be nice to find a direct expression for M_n. For this, we want to look at the problem a bit differently, namely, we will look at how the intersections are distributed throughout the maze. Note that there are $n+1$ exits if there are n intersections in the maze. If we look from the first intersection, to the left there are $0 \leq k \leq n-1$ intersections and to the right $n-1-k$. Let I_n denote the number of mazes with n intersections (so with $n+1$ exits). We see that:

$$I_n = \sum_{k=0}^{n-1} I_k I_{n-1-k}.$$

One might recognise this formula as the recursive formula for the Catalan numbers C_n. We know that $C_n = \frac{1}{n+1}\binom{2n}{n}$. Thus we can derive a direct formula for M_n:

$$M_n = I_{n-1} = C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}.$$

This formula is more direct, but note that to calculate a binomial coefficient, your program will still use recursion.

If you do not recognise this, we can also find this relation by using the hint and looking at the generating function with I_n as coefficients:

$$f(x) = \sum_{n=0}^{\infty} I_n x^n.$$

If we write out the recurrence $I_n = \sum_{k=0}^{n-1} I_k I_{n-1-k}$ that we have found in this function, we can see that we get

$$f(x) = 1 + x f(x)^2.$$

If we see this last equation as a quadratic function, we get a solution:

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Now as we know that $I_0 = 1$, we must have that $f(0) = 1$ and therefore that $f(x) = \frac{1-\sqrt{1-4x}}{2x}$. Now to find a direct formula for the I_n, we want to split this expression into $\sum_{n=0}^{\infty} a_n x^n$ for some a_n. To do this, we will take a Taylor approximation of $f(x)$. First, let us look at the Taylor expansion of $\sqrt{1+y}$:

$$\sqrt{1+y} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} y^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^n(2n-1)} \binom{2n}{n} y^n.$$

Then we see that:

$$\sqrt{1-4x} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^n(2n-1)} \binom{2n}{n} (-4x)^n = \sum_{n=0}^{\infty} \frac{-1}{2n-1} \binom{2n}{n} x^n.$$

Note that the term of $n = 0$ is equal to 1. Then we see that:

$$f(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{\sum_{n=1}^{\infty} \frac{-1}{2n-1} \binom{2n}{n} x^n}{2x} = \sum_{n=0}^{\infty} \frac{-1}{2(2n-1)} \binom{2n}{n} x^{n-1}$$

Now doing a change of variable $i = n - 1$ gives us:

$$f(x) = \sum_{i=0}^{\infty} \frac{-1}{2(2i+1)} \binom{2i+2}{i+1} x^i.$$

Therefore we have now found a direct formula for I_n:

$$M_{n+1} = I_n = \frac{-1}{2(2n+1)} \binom{2n+2}{n+1}.$$

One can rewrite this answer to the one we saw using Catalan numbers to see that they coincide.